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SOLUTION SKETCHES TO HOMEWORK 3

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Homework 3.1 (Construction in the proof of Mittag-Leffler's theorem*)**.** Construct explicitly (in the sense of an explicit series) a holomorphic function $f\colon \mathbf{C} \setminus \{\sqrt{n} : n \in \mathbf{N}\} \to \mathbf{C}$ such that for every $n \in \mathbf{N}$, the function f has the principal part

$$q_n(z) := \frac{\sqrt{n}}{z - \sqrt{n}}$$

at $z_n := \sqrt{n}$[1].

Homework 3.2 (Partial fraction decomposition of $\pi^2/\sin^2(\pi z)$)**.** The goal of this exercise is to prove the formula

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n\in\mathbf{Z}} \frac{1}{(z-n)^2}.$$

This will be achieved in several steps. Keep in mind the formula

$$\sin(z) = \frac{1}{2\mathrm{i}}\big[\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}\big].$$

a. Show the assignment $f(z) := \pi^2/\sin^2(\pi z)$ has its singularities exactly in $\mathbf{Z}$ and determine the principle parts of the Laurent series expansion in those points.
b. Show the series

$$g(z) := \sum_{n\in\mathbf{Z}} \frac{1}{(z-n)^2}$$

converges locally uniformly on $\mathbf{C} \setminus \mathbf{Z}$, so that it is meromorphic. Conclude the difference $g(z) - f(z)$ can be extended to an entire function.
c. Show $f(z)$ converges to zero as $|\Im z| \to \infty$ uniformly in $\Re z$.
d. Show the same statement for the function g. Then prove that the difference $g - f$ is bounded on $\mathbf{C}$ and conclude the proof[2].

Solution. a. The function f has singularities exactly in all zeros of the assignment $\sin(\pi z)$. Note that $\sin(\pi z) = 0$ for every $z \in \mathbf{Z}$.

We now argue that these are the only zeros. Write $z = x + \mathrm{i}\,y$, where $x, y \in \mathbf{R}$. Then

$$\sin(z) = \frac{1}{2\mathrm{i}}\big[\mathrm{e}^{\mathrm{i}x}\,\mathrm{e}^{-y} - \mathrm{e}^{-\mathrm{i}x}\,\mathrm{e}^{y}\big],$$

so that $\sin(z) = 0$ implies $\mathrm{e}^{2\mathrm{i}x} = \mathrm{e}^{2y}$. The left hand side has modulus one, which yields $e^{2y} = 1$. By injectivity of the real-valued exponential function, we infer $y = 0$. In turn, from the previous identities of exponential functions, we obtain $x \in \pi\mathbf{Z}$.

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[1]**Hint.** Prove that a second order Taylor polynomial p_n of q_n yields the local normal convergence of the sum $f = \sum_{n\in\mathbf{N}}[q_n - p_n]$, cf. Theorem 2.2.

[2]**Hint.** Note that $(g - f)(z+1) = (g - f)(z)$ for every $z \in \mathbf{C}$.

In order to determine the principal part in a singularity $z^* \in \mathbf{Z}$ we first treat the case $z^* = 0$. Since the assignment $\sin(z)/z$ has a removable singularity in $z^* = 0$, it follows that f has a pole of second order in $z^* = 0$. Hence its Laurent series takes the form

$$f(z) = a_{-2}z^{-2} + a_{-1}z^{-1} + \sum_{n=0}^{\infty} a_n z^n.$$

Since the corresponding integrals for the coefficients are quite difficult to evaluate, we use this structure to determine the coefficients a_{-1} and a_{-2}. Note that

$$a_{-2} = \lim_{z\to 0} z^2 f(z) = \lim_{z\to 0} \frac{(\pi z)^2}{\sin^2(\pi z)} = 1.$$

Next, for the coefficient a_{-1} we use $\sin(z)/z \to 1$ as $z \to 0$ to calculate

$$\begin{aligned} a_{-1} &= \frac{\mathrm{d}}{\mathrm{d}z}\bigg|_0 z^2 f(z) \\ &= \lim_{z\to 0} \frac{2\pi^2 z \sin^2(\pi z) - 2(\pi z)^2 \sin(\pi z)\cos(\pi z)\pi}{\sin^4(\pi z)} \\ &= \lim_{z\to 0} \frac{2\pi - 2\pi\cos(\pi z)}{\sin(\pi z)} \\ &= 0; \end{aligned}$$

here, we have used $\cos(z) = 1 - z^2 + \mathrm{O}(z^4)$ as $z \to 0$. Hence the principal part at $z^* = 0$ reads $q(z) := 1/z^2$.

To treat the other singularities we note that $f(z+1) = f(z)$ for all $z \in \mathbf{C} \setminus \mathbf{Z}$. Indeed, write again $z = x + \mathrm{i}\,y$, where $x, y \in \mathbf{R}$. Then

$$\sin(\pi z + \pi) = \frac{1}{2\mathrm{i}}\left[\mathrm{e}^{\mathrm{i}\pi(x+1)}\,\mathrm{e}^{-y} - \mathrm{e}^{-\mathrm{i}\pi(x+1)}\,\mathrm{e}^{y}\right] = -\sin(\pi z),$$

so that taking the inverse square yields the claim. By this periodicity property it follows that the coefficients of the Laurent series are also periodic. Hence at a general $z_n \in \mathbf{Z}$ the principal part is given by $q_n(z) := 1/(z - z_n)^2$.

b. Let $K \subset \mathbf{C} \setminus \mathbf{Z}$ be compact. Then there exists a constant $c(K)$ with $\sup_{z\in K} |z| \le c(K)$. In particular, for every $n \in \mathbf{N}$ with $|n| \ge 2c(K)$ and every $z \in K$, the triangle inequality implies $|z - n| \ge |n| - c(K) \ge |n|/2$, which gives

$$\sum_{|n|\ge 2c(K)} \sup_{z\in K} \left|\frac{1}{(z-n)^2}\right| \le \sum_{|n|\ge 2c(K)} \frac{4}{n^2} < \infty.$$

Hence the series g converges locally normally on $\mathbf{C} \setminus \mathbf{Z}$. By Lemma 1.11 it follows that g is holomorphic on $\mathbf{C} \setminus \mathbf{Z}$. Since all singularities are isolated and poles of second order, we deduce that g is meromorphic on $\mathbf{C}$. As its principal parts agree with the ones of f in all singularities, it follows from Remark 2.3 that the difference $g - f$ can be extended to an entire function.

c. We show the function $|\sin(x + \mathrm{i}y)|$ blows up when $|y| \to +\infty$ uniformly in $x \in \mathbf{R}$. Indeed, by the triangle inequality we have

$$|\sin(x + \mathrm{i}y)| = \left|\frac{1}{2\mathrm{i}}\left[\mathrm{e}^{\mathrm{i}x}\,\mathrm{e}^{-y} - \mathrm{e}^{-\mathrm{i}x}\,\mathrm{e}^{y}\right]\right| \ge \frac{1}{2}\left[\mathrm{e}^{|y|} - \mathrm{e}^{-|y|}\right].$$

The right hand side goes to ∞ as $|y| \to \infty$ uniformly in $x \in \mathbf{R}$. This proves the claim.

d. Fix $n \in \mathbf{Z}$ and write $z = x + \mathrm{i}y$, where $x, y \in \mathbf{R}$. By the periodicity $g(z+1) = g(z)$ for all $z \in \mathbf{C} \setminus \mathbf{Z}$ (which follows by an index shift) it suffices to take $x \in [0, 1]$. Using the equivalence of the ℓ^1-norm and the usual euclidean (viz. ℓ^2-)norm on $\mathbf{R}^2$ we deduce there exists a constant $c > 0$ such that

$$|z-n| \geq \frac{1}{c}(|y|+|x-n|) \geq \begin{cases} \frac{1}{c}\left[|y|+|n|-1\right] & \text{if } n \geq 1, \\ \frac{1}{c}\left[|y|+|n|\right] & \text{if } n \leq 0. \end{cases}$$

Hence, performing further index shifts, we infer that

$$\begin{aligned} \Big|\sum_{n\in\mathbf{Z}} \frac{1}{(z-n)^2}\Big| &\leq c^2 \sum_{n\geq 1} \frac{1}{(|y|+|n|-1)^2} + c^2 \sum_{n\leq 0} \frac{1}{(|y|+|n|)^2} \\ &\leq 2c^2 \sum_{n\geq 0} \frac{1}{(|y|+|n|)^2} \\ &\leq 2c^2 \sum_{n\geq |y|-1} \frac{1}{n^2}. \end{aligned}$$

Clearly the last term vanishes as $|y| \to +\infty$.

It remains to show the difference $g - f$ is bounded on $\mathbf{C}$. We already know that it is holomorphic and periodic in the sense that $(g-f)(z+1) = (g-f)(z)$ for every $z \in \mathbf{C}$. Hence on each strip of the form $\mathbf{R} \times [-a, a]$, where $a > 0$, it is bounded. From c. and the previous discussion we further know that there exists some $a^* > 0$ such that for all $z = x + \mathrm{i}y$ with $x \in \mathbf{R}$ and $y \in \mathbf{R}$ with $|y| > a^*$ it holds that $|(g-f)(z)| \leq 1$. Hence $g - f$ is a bounded, entire function. By Liouville's theorem it is therefore constant and again by c. and the previous discussion it follows $f = g$.

Homework 3.3 (Weierstraß elliptic function)**.** Let $\omega_1, \omega_2 \in \mathbf{C}$ be $\mathbf{R}$-linearly independent. Show that up to an additive constant there exists one and only one holomorphic function[3] $\wp\colon \mathbf{C} \setminus \{m\omega_1 + n\omega_2 : m, n \in \mathbf{Z}\} \to \mathbf{C}$ such that

a. $\wp$ has principal part $q_1(z) := 1/z^2$ in $d_1 := 0$ and
b. $\wp$ is $\{\omega_1, \omega_2\}$-periodic, i.e. for every $z \in \mathbf{C} \setminus \{m\omega_1 + n\omega_2 : m, n \in \mathbf{Z}\}$,

$$\begin{aligned} \wp(z+\omega_1) &= \wp(z), \\ \wp(z+\omega_2) &= \wp(z). \end{aligned}$$

You can use without proof that

$$\sum_{\substack{n,m\in\mathbf{Z} \\ |n|+|m|\neq 0}} \frac{1}{(n^2+m^2)^{3/2}} < \infty.$$

Solution. We first show the uniqueness statement. Suppose that there are two functions $\wp_1$ and $\wp_2$ with the stated properties. Then the difference $g := \wp_1 - \wp_2$ has a removable singularity in 0 and by periodicity also in each point $z_{n,m} = m\omega_1 + n\omega_2$, where $n, m \in \mathbf{Z}$. We conclude that g can be extended to an entire function. Since the set $\{\omega_1, \omega_2\}$ forms a basis of $\mathbf{C}$ with respect to field $\mathbf{R}$, the local boundedness of (the extended version of) g and the periodcity imply that g is bounded on $\mathbf{C}$. Hence g is constant by Liouville's theorem. This shows the uniqueness claim.

Next we construct the function $\wp$. The basic idea is to use the structure given by the Mittag-Leffler theorem, that means we consider the countable set $S := \{z_{n,m} = m\omega_1 + n\omega_2 : m, n \in \mathbf{Z}\}$ and the corresponding principal parts $q_{n,m}(z) := 1/(z - z_{n,m})^2$. The difficult part is how to choose the polynomials in order to ensure periodicity. Non-constant polynomials are never periodic. Hence we try to construct constant ones. The zeroth-order expansion of

[3] If we require that the zeroth order coefficient of the Laurent series in the origin vanishes, this function is called the Weierstraß $\wp$-function.

$q_{n,m}$ in zero is given by $q_{n,m}(0) = 1/z_{n,m}^2$ whenever $n^2 + m^2 \neq 0$. Thus our ansatz is

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{n,m\in\mathbf{Z},\\ n^2+m^2\neq 0}} \Big[\frac{1}{(z-z_{n,m})^2} - \frac{1}{z_{n,m}^2}\Big].$$

We show this series converges locally normally on $\mathbf{C} \setminus \{z_{n,m} : m,n \in \mathbf{Z}\}$. Then it satisfies all claimed properties — the periodicity from b. following by an index shift. Let $K \subset \mathbf{C} \setminus \{z_{n,m} : m,n \in \mathbf{Z}\}$ be compact. For all $z \in K$, algebraic manipulations give

$$\Big|\frac{1}{(z-z_{n,m})^2} - \frac{1}{z_{n,m}^2}\Big| \leq \frac{|z^2 - 2z_{n,m}z|}{|z_{n,m}|^2\,|z-z_{n,m}|^2}. \tag{3.1}$$

Since K is bounded, the quantity $C_K := \sup_{z\in K}|z|$ is finite. Next note that the $\mathbf{R}$-linear mapping Ω defined through $\Omega(1) := \omega_1$ and $\Omega(\mathrm{i}) := \omega_2$ can be interpreted as an invertible, $\mathbf{R}$-linear map from $\mathbf{C}$ to $\mathbf{C}$. Thus there exists a constant $c(\omega_1,\omega_2)$ such that

$$|z_{n,m}| = |\Omega(n+mi)| \geq c(\omega_1,\omega_2)\,|n+mi| = c(\omega_1,\omega_2)\sqrt{n^2+m^2}. \tag{3.2}$$

Hence there exists $n(K) \in \mathbf{N}$ such that for every $n,m \in \mathbf{Z}$ with $n^2+m^2 \geq n(K)$,

- $|z_{n,m}| \geq C_K$ and
- $|z_{n,m}| - C_K \geq |z_{n,m}|/2$.

The second item implies that $|z - z_{n,m}| \geq |z_{n,m}|/2$ for every $z \in K$. Inserting these bounds and (3.2) in (3.1) yields that for $z \in K$ and some large constant C,

$$\begin{aligned}\Big|\frac{1}{(z-z_{n,m})^2} - \frac{1}{z_{n,m}^2}\Big| &\leq \frac{|z|^2 + 2|z_{n,m}||z|}{|z_{n,m}|^2(1/2|z_{n,m}|)^2}\\ &\leq \frac{4C_K^2 + 8|z_{n,m}|C_K}{|z_{n,m}|^4}\\ &\leq \frac{12C_K}{|z_{n,m}|^3}\\ &\leq \frac{C}{(n^2+m^2)^{3/2}}.\end{aligned}$$

According to the hint[4], we have

$$\sum_{\substack{n,m\in\mathbf{Z}\\ n^2+m^2\geq n(K)}} \frac{1}{(n^2+m^2)^{3/2}} < \infty.$$

From the previous considerations,

$$\sum_{\substack{n,m\in\mathbf{Z}\\ n^2+m^2\geq n(K)}} \sup_{z\in K}\Big|\frac{1}{(z-z_{n,m})^2} - \frac{1}{z_{n,m}^2}\Big| < \infty.$$

We conclude by local normal convergence that $\wp$ satisfies all the desired properties.

[4]This can be shown via comparison to an integral.